[This question paper contains 4 printed pages.]

13 Your Roll No. 2023

Sr. No. of Question Paper : 4548

Unique Paper Code	:	32351201
Name of the Paper	:	Real Analysis (CBCS-LOCF)
Name of the Course	:	**B.Sc. (Hons) Mathematics**
Semester	:	II

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any two parts from each question.

1\. (a) If x and y are positive real numbers with $x < y$, then prove that there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$. (6.5)

(b) Define Infimum and Supremum of a nonempty set of $\mathbb{R}$. Find infimum and supremum of the set

$$S = \left\{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}.$$ (6.5)

P.T.O.

(c) State the completeness property of $\mathbb{R}$, hence show that every nonempty set of real numbers which is bounded below, has an infimum in $\mathbb{R}$. (6.5)

2. (a) Prove that there does not exist a rational number $r \in \mathbb{Q}$ such that $r^2 = 2$. (6)

(b) Define an open set and a closed set in $\mathbb{R}$. Show that if $a, b \in \mathbb{R}$ then the open interval (a, b) is an open set. (6)

(c) Let S be a nonempty bounded set in $\mathbb{R}$. Let $a > 0$, and let $aS = \{as: s \in S\}$. Prove that $\inf(aS) = a(\inf S)$ and $\sup(aS) = a(\sup S)$. (6)

3. (a) Define limit of a sequence. Using definition show that $\lim_{n\to\infty}\left(\frac{3n+1}{2n+5}\right) = \frac{3}{2}$. (6.5)

(b) Prove that every convergent sequence is bounded. Is the converse true? Justify. (6.5)

(c) Let $x_1 = 1$ and $x_{n+1} = \frac{1}{4}(2x_n + 3)$ for $n \in \mathbb{N}$. Show that $\langle x_n \rangle$ is bounded and monotone. Find the limit. (6.5)

4. (a) If $\langle a_n \rangle$ and $\langle b_n \rangle$ converges to a and b respectively, prove that $\langle a_n b_n \rangle$ converges to ab. (6)

(b) Show that $\lim_{n \to \infty} n^{1/n} = 1$. (6)

(c) State Cauchy Convergence Criterion for sequences. Hence show that the sequence $\langle a_n \rangle$, defined by $a_n = 1 + \frac{1}{2} + \cdots\cdots + \frac{1}{n}$, does not converge. (6)

5. (a) Prove that if an infinite series $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n \to \infty} a_n = 0$. Hence examine the convergence of $\sum_{n=1}^{\infty} \frac{n}{2n+3}$. (6.5)

(b) Examine the convergence or divergence of the following series.

(i) $\frac{2}{5} + \frac{4}{8} + \frac{6}{11} + \cdots$ (6.5)

(ii) $\sum_{n=1}^{\infty}\left(\frac{3n+5}{2n+1}\right)^{n/2}$

(c) Prove that $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^{p}}$, $p>0$ is convergent for $p>1$ and divergent for $p\leq 1$. (6.5)

6. (a) State and prove ratio test (limit form). (6)

(b) Examine the convergence or divergence of the following series. (6)

(i) $\sum_{n=1}^{\infty}\frac{n^{3}+1}{n^{4}+3n^{2}+2n}$

(ii) $3+\frac{3^{2}}{2!}+\frac{3^{3}}{3!}+\frac{3^{4}}{4!}+\cdots$

(c) Prove that the series $\frac{1}{\ln 2}-\frac{1}{\ln 3}+\frac{1}{\ln 4}-\frac{1}{\ln 5}+\cdots$ is conditionally convergent. (6)

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Your Roll No. 2023

Sr. No. of Question Paper : 4668 E

Unique Paper Code : 32351202

Name of the Paper : Differential Equations

Name of the Course : B.Sc. (Hons.) Mathematics

Semester : II

Duration : 3 Hours

Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Use of non-programmable scientific calculator is allowed.

Section – 1

1. Attempt any **three** parts. Each part is of **5** marks.

(a) Solve the initial value problem

$$(2y\sin x\cos x + y^2\sin x)dx + (\sin^2 x - 2y\cos x)dy = 0, \qquad y(0) = 3$$

(b) Solve the differential equation

$$(x^2 - 3y^2)dx + 2xy\,dy = 0$$

(c) Solve the differential equation

$$xy'' + 2y' = 6x$$

(d) Solve the differential equation

$$(x + 2y + 3)dx + (2x + 4y - 1)dy = 0$$

2. Attempt any **two** parts. Each part is of **5** marks.

(a) In a certain culture of bacteria, the number of bacteria increased sixfold in 10 hours. How long did it take for the population to double?

(b) An arrow is shot straight upward from the ground with an initial velocity of 160 ft/s. It experiences both the deceleration of gravity and deceleration $v^2/800$ due to air resistance. How high in the air does it go?

(c) A cake is removed from an oven at 210°F and left to cool at room temperature, which is 70°F. After 30 min the temperature of the cake is 140°F. When will it be 100°F?

Section – 2

3. Attempt any two parts. Each part is of **8** marks.

(a) The following differential equation describes the level of pollution in the lake

$$\frac{dC}{dt} = \frac{F}{V}(C_{in} - C),$$

where V is the volume, F is the flow (in and out), C is the concentration of pollution at time t and C_{in} is the concentration of pollution entering the lake. Let $V = 28 \times 10^6 m^3$, $F = 4 \times 10^6 m^3/month$. If only fresh water enters the lake,

i. How long would it take for the lake with pollution concentration $10^7 parts/m^3$ to drop below the safety threshold$4 \times 10^6 parts/m^3$?

ii. How long will it take to reduce the pollution level to 5% of its current level?

(b) In view of the potentially disastrous effects of overfishing causing a population to become extinct, some governments impose quotes which vary depending on estimates of the population at the current time. One harvesting model that takes this into account is

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right) - h_0 X.$$

i. Show that the only non-zero equilibrium population is

$$X_c = K\left(1 - \frac{h}{r}\right).$$

ii. At what critical harvesting rate can extinction occur?

(c) In a simple battle model, suppose that soldiers from the red army are visible to the blue army, but soldiers from the blue army are hidden. Thus, all the red army can do is fire randomly into an area and hope they hit something. The blue army uses aimed fire.

i. Write down appropriate word equations describing the rate of change of the number of soldiers in each of the armies.

ii. By making appropriate assumptions, obtain two coupled differential equations describing this system.

iii. Write down a formula for the probability of a single bullet fired from a single red soldier wounding a blue soldier in terms of the total area A and the area exposed by a single blue soldier A_b.

iv. Hence write the rate of wounding of both armies terms of the probability and the firing rate.

Section – 3

4. Attempt any three parts. Each part is of **6** marks.

(a) Find the general solution of the differential

$$x^3y''' + 6x^2y'' + 4xy' = 0.$$

(b) Using the method of undetermined coefficients, solve the differential equation

$$y''' - 2y'' + y' = 1 + xe^x,\ y(0) = y'(0) = y''(0) = 1.$$

(c) Using the method of Variation of parameters, solve the differential equation

$$y'' + 3y' + 2y = 4e^x.$$

(d) Show that $y_1 = 1$ and $y_2 = \sqrt{x}$ are solutions of

$$yy'' + (y')^2 = 0,$$

but the sum $y = y_1 + y_2$ is not a solution. Explain why?

Section – 4

5. Attempt any **two** parts. Each part is of **8** marks.

(a) The pair of differential equations

$$\frac{dP}{dt} = rP - \gamma PT, \frac{dT}{dt} = qP,$$

where r, γ and q are positive constants, is a model for a population of microorganisms P, which produces toxins T which kill the microorganisms.

i. Given that initially there are no toxins and P_0 microorganisms, obtain an expression relating the population density and the amount of toxins.

ii. Hence, give a sketch of a typical phase-plane trajectory.

iii. Using phase-plane trajectory, describe what happens to the microorganisms over time.

(b) A model of a three species interaction is :

$$\frac{dX}{dt} = a_1X - b_1XY - c_1XZ,$$
$$\frac{dY}{dt} = a_2XY - b_2Y,$$
$$\frac{dZ}{dt} = a_3XZ - b_3Z$$

Where a_i, b_i, c_i for $i = 1, 2, 3$ are all positive constants. Here $X(t)$ is the prey density and $Y(t)$ and $Z(t)$ are the two predator species densities.

i. Find all possible equilibrium populations. Is it possible for the three populations to coexist in equilibrium ?

ii. What does this suggest about introducing an additional predator into an ecosystem ?

(c) In a long range battle, neither army can see the other, but fires into a given area. A simple mathematical model describing this battle is given by the coupled differential equations

$$\frac{dR}{dt} = -c_1RB, \qquad \frac{dB}{dt} = -c_2RB$$

where c_1 and c_2 are positive constants.

i. Use the chain rule to find a relationship between R and B, given the initial numbers of soldiers for the two armies are r_0 and b_0, respectively.

ii. Draw a sketch of typical phase-plane trajectories.

iii. Explain how to estimate the parameter c_1 given that the blue army fires into a region of area A.